Sedimentation of Flocculent Suspensions: State of the Art

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Three modes or types of sedimentation are recognized in flocculent suspensions: *Clarification,* in which floccules are separated and settle independently; *zone settling,* in which floccules are incorporated into some solids structure so that all are constrained to subside at more or less the same rate; and *compression* or *compaction,* in which the solids structure is strong enough to exhibit a compressive yield value. Current models for sedimentation in the three modes are reviewed, with particular emphasis on their use and reliability (or lack of it) for sedimentation basin design. Most models of thickening presented in the literature derive from a basic partial differential equation for force balance. Models differ in which terms of the equation are disregarded. Several of the theories of compaction are essentially equivalent solutions for the same model, but with different sets of independent variables.

REGIMES OF SEDIMENTATION

Sedimentation is here defined as the entire process by which particles settle out of fluid suspension. All steps of the process are profoundly affected by interfacial phenomena. Van der Waals or London forces, if not overpowered by electrostatic repulsions (zeta potentials) or fluid-dynamic shear, cause particles to cohere on contact. This gives rise to three different modes or regimes of sedimentation. Which one is followed depends upon the solids concentration, and on the relative tendency of the particles to cohere. The factors are important because they correspond to different sedimentation models or mechanisms.

Types of sedimentation are represented paragenetically in Figure 1. The left side represents particles with little tendency to cohere; the right side, those for which interparticle cohesion is strong. The vertical axis represents particle concentration, with more concentrated suspensions at the top.

At low concentrations, there occurs a regime called *clarification.* Particles are, on the average, far apart and free to settle independently, but collisions occur. If the particles then cohere, they grow into clumps, or floccules, whose settling rate increases as they grow. If, on the other hand, they do not cohere, each bumps downward at its characteristic rate. Thus there are two types of settling in clarification: *flocculent,* and *particulate.* There is no sharp boundary between the two; one grades gradually into the others. And, it is common practice to modify behavior in this regime by adding flocculants to aid clarification.

Clarification behavior is easily recognized in a batch test. Slower settling particles string out behind faster settling ones. The upper layers gradually thin out, or clarify. Settled solids collect at the bottom in a layer whose upper boundary rises as solids settle into it.

As particles become more concentrated, crowded closer together, they finally reach a point where each is in contact with others. If they then have any tendency to cohere, they link into some sort of floc structure. Whether the structure formed is continuous (Fitch 1962, 1972) or consists of a bed of closely-spaced floccules (Roberts 1949) is a moot question. But in any case, particles held in the structure are constrained to settle at the same rate. Solids subside with a sharp interface between pulp and supernatant. Slurry enters a *zone-settling* regime and exhibits *line-settling* behavior. This is clearly distinguished from clarification behavior in a batch-settling test, because the visible interface moves downward from the top, rather than building upward from the bottom as in clarification.

As concentration is further increased, the pulp structure becomes so firm it develops compressive strength. Each layer of solids is able to transmit mechanical support to layers above. In order to compact the structure to a higher solids concentration, a solids stress or squeeze must be exerted on it. In such a condition, the pulp is said to be in compression and the regime is called *compression,* or *compaction.*

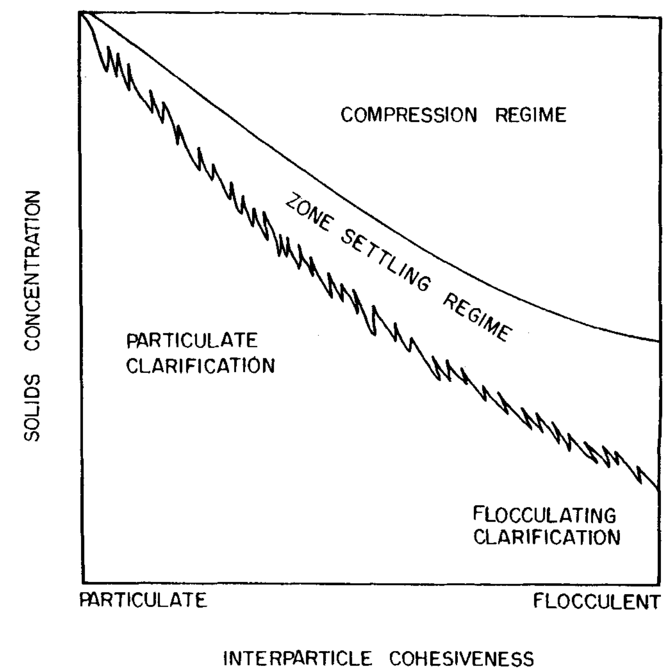

Figure 1. Regimes of sedimentation.

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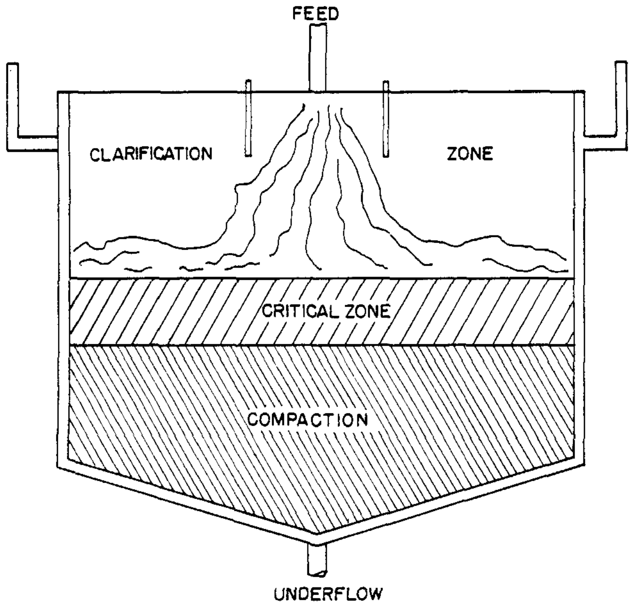

Figure 2. Sedimentation zones in continuous thickening.

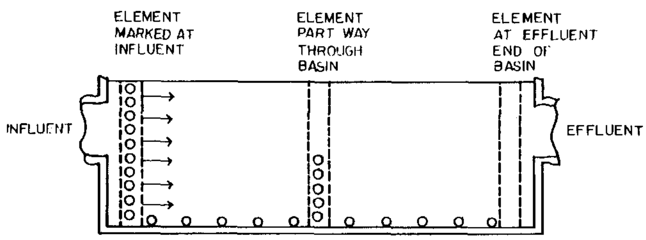

Figure 3. Ideal clarification basin (after Hazen).

In addition to the three regimes described, concentrated pulps also exhibit short-circuiting, or channeling. If we view the solid structure as a porous medium, channeling increases its permeability and lowers its resistance to percolation of fluid. As a result, settling rates are augmented, sometimes many-fold. Once channeling was considered a concomitant and characteristic of compression. If you saw channeling, you knew the pulp was in compression. At this time we are not sure exactly what causes channeling, or of its concentration bounds. It is being studied. And, there may be more than one sort. A term coined to cover any channeling or local segregation of the pulp into a grossly non-homogeneous structure is "phase-settling."

All three regimes may not be present in any continuous sedimentation pool. But if they are, they would ideally be distributed as shown in Figure 2. Feed suspension, which carries settleable solids and hence must have a higher specific gravity than clarified liquid, plunges through supernatant layers and spreads out at its level of hydrostatic equilibrium (Sawyer 1956, Roberts and Fitch 1956, Fitch and Lutz 1960, Scott 1973). In this feed layer, liquid destined for the overflow separates, carrying fines that do not settle out. If these fines are not flocculent, they are swept into the overflow. If they are, they may flocculate during the time the liquid is rising to the overflow. Floccules thus formed can settle back into and through the feed layer. Or under some conditions, they may pile up as a slime layer above the feed zone.

Solids settling out of or through the feed layer pass into a critical zone, if one exists. Critical zones (to be discussed later), form when the flux of solids fed to a thickener exceeds that which can settle through a most limiting concentration. The excess solids add to the critical zone, increasing its depth until it fills the thickener and the excess overflows. Critical zones form only when the solids settling capacity of the pool is exceeded, and should not be present in normal operation. One of the design criteria is to provide enough pool area so that a critical zone will not form at specified maximum solids throughput. In classical Coe and Clevenger theory, critical zones are conceived to form only in the zone settling regime. However, it appears that they can form also in the compression regime (Fitch 1966) and Dixon (1977) now contends that they can form only in the compression regime.

Solids next enter a compaction zone. As they pass downward through it they are subjected to ever-increasing squeeze or solids stress. If they are compressible, they are compacted to an ever higher concentration. Note, however, that the solids are still settling with respect to liquid. And, it turns out that most of their weight is sustained hydrodynamically. Only a fraction of it is available to produce solids stress. Thus the solids stress and concentration reached at the bottom of a compaction zone is not an obvious function of solids loading and zone depth.

Flow patterns in an actual settler may be quite different than that shown in Figure 2. But on the basis of this idealized model, clarification can be identified with what happens between the feed layer and the overflow. It is governed primarily by the velocity at which the liquid moves upward to the overflow (liquid flux, overflow rate) and by the retention of the fluid in the clarification zone. Thickening relates to what happens between feed layer and underflow. It is governed by the downward flux of solids, and by depth of the compaction zone.

CLARIFICATION

The art of clarification comprises many things: The chemistry of flocculation, the kinetics of coagulation, the settling behavior of floccules, the hydraulic behavior of settling tanks, and the interactions between these factors. There is an enormous relevant literature, largely in publications concerned with colloid science, and with sanitary or environmental engineering. It would go far beyond the scope of one article to review the status of this theory. This discussion, therefore, will be limited to the concepts or models most directly underlying current engineering practice.

One basic concept is "the ideal sedimentation basin." A relationship between batch and continuous operation in the clarification regime is developed with an ideal horizontal flow model introduced by Hazen (1904). His ideal basin is rectagular and of uniform depth (Figure 3). Suspension flows uniformly from one end to the other. That is, all elements of flow have identical velocity vectors.

If a cylindrical element of suspension, extending from top to bottom of the pool is identified at the inlet, it will maintain its shape and identity as it traverses the pool. The clarification attained in the pool is identically that attained in the element, just as it reaches the pool discharge. The same reasoning would apply to an inclined settling tube or lamina, if the flow were ideal (Figure 4). So to predict the performance of such an ideal basin, one needs only a batch settling test in a cylinder or long tube having a height equal to the depth of the basin, allowing it to settle for a time equal to the pool retention time.

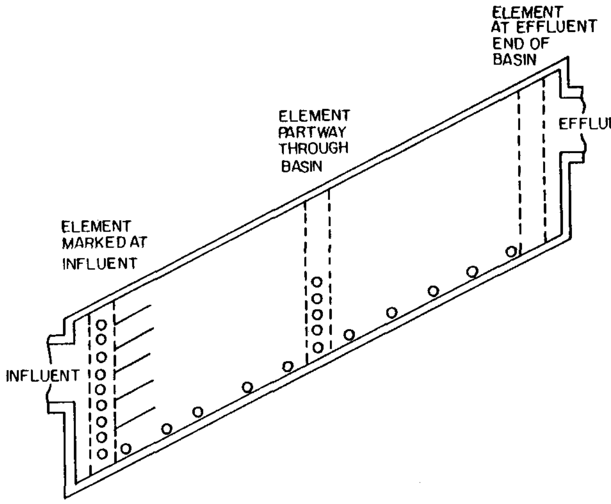

Figure 4. Ideal inclined settling chamber.

Actual pools are far from ideal because of turbulence and non-uniform flow distribution. This is allowed for in calculations by empirical efficiency factors, one applied to retention time, the other to overflow rate. For most practical basins, retention efficiency is low. A typical figure for basins of what is considered good design might be about 33%. In other words, the actual basin must have three times the nominal retention time of an ideal basin to achieve corresponding results. There is room for improvement.

Long-tube test results, combined with empirical efficiencies for a given basin, yield design predictions that are fairly close. That is, the errors in prediction seem not greater than deviations caused by the variability of the feed suspensions concerned. There are, however, some simpler procedures for special cases.

Clarification may be considered a two-step process, analogous to a two-step chemical reaction. First the particles flocculate, then the floccules settle out. Flocculation takes time, settling takes area. And in a great many practical cases, one step or the other will be rate-dominating.

If flocculation takes place very rapidly, or if the suspension is pre-flocculated before being introduced into the settling basin, then the floccules have the same settling rate over essentially their entire sedimentation time. Essentially particulate clarification behavior is approached. In such cases, basin area and not basin retention time will govern solids removals. And, of particular practical consequence, basin retention efficiency ceases to be a factor in the clarification results. Basin design for particulate clarification is therefore less demanding than for flocculent clarification.

The fact that basin depth, and hence basin retention, is not a factor in particulate sedimentation has classically been demonstrated for ideal basins (Camp 1946, 1953). A more general proof, not assuming ideal flow but allowing flow vectors to vary with depth, has been published (Fitch 1956). A simpler one follows:

The locus of topmost particles of any given size class in a sedimentation basin at steady state will be some surface S. If everywhere at this surface the particles have a settling rate **u** with respect to suspension in their neighborhoods, then there will be a vertical component of fluid flux, or fluid phase velocity, past this surface equal to u. The release of supernatant through an element of surface, dS, must then be:

$$dQ_o = \mathbf{u} \cdot d\mathbf{S} = u dA \quad (1)$$

By integration over the settling area

$$Q_o = uA \quad (2)$$

Thus the amount of clarified overflow Q_o that can be produced is a function of pool area and the settling rate u of the particles. Pool depth, and hence pool retention time, does not enter. And overall solids removals will be governed by overflow rate v_o, which is Q_o/A.

For many years it was argued in the sanitary engineering literature that removals in flocculant clarification should also be independent of retention time. The arguments were valid deductions from their premises, but were based on a tacit assumption that was false. Although the unsoundness of such arguments has been demonstrated (Fitch 1957, 1958) they are still sometimes advanced, particularly in promotional literature for lamina-type clarifiers.

The other special case arises when flocculation is slow compared to sedimentation. In such cases, it is basin retention time, and not overflow area per se, that is important.

It appears that once a floccule starts to grow, it continues to grow quite rapidly to a large size, and settles out. Within certain limits, then, if flocculation is slow, solids removal in a clarification basin will be governed by flocculation kinetics and the effective basin retention time. Removals from an ideal basin can be predicted from simple retention tests, in which residual, unsettled solids concentration in a batch of suspension is measured as a function of time. The retention efficiency of a basin can be predicted from its hydraulic behavior by the methods of reaction engineering (Fiedler and Fitch 1959).

Many studies have been made of flocculation kinetics. The following have had the most bearing on existing design practice: Smoluchowski (1916, 1917), Camp and Stein (1946), Fiedler and Fitch (1959). These show flocculation to be a second-order reaction:

$$\frac{dC}{dt} = -KC^2 \quad (3)$$

In integrated form

$$K(t - t_0) = \frac{1}{C - C_\infty} - \frac{1}{C_0 - C_\infty} \quad (4)$$

For high removals, $(C - C_\infty) << (C_0 - C_\infty)$. As an approximation, $1/(C_0 - C_\infty)$ can therefore be neglected and:

$$K(t - t_0) \cong \frac{1}{C_0 - C_\infty} \quad (5)$$

or

$$C \cong \frac{1}{Kt} + C_0 \quad (6)$$

The value of C_∞, which is the concentration of solids present that do not enter into the flocculation reaction, can therefore be determined by plotting C vs $1/t$. Points should lie along a straight line whose slope measures $1/k$, and whose C-intercept measures C_∞. But with the important exception of sewage sludges, with their "unsettleable solids," C_∞ appears usually to be zero.

When C_∞ is zero, a plot of $1/C$ vs. t should be linear, with a $1/C$ intercept equal to $1/C_0$: Empirically, plots of the reciprocal of solids concentration remaining in suspension after any given time, vs. time, turn out usually to be linear, as predicted by theory. But the lines often have a $1/C$ intercept below $1/C_0$. The flocculation reaction seems to start slowly. A reasonable theoretical explanation of such behavior as given by Fiedler and Willus (1973), follows:

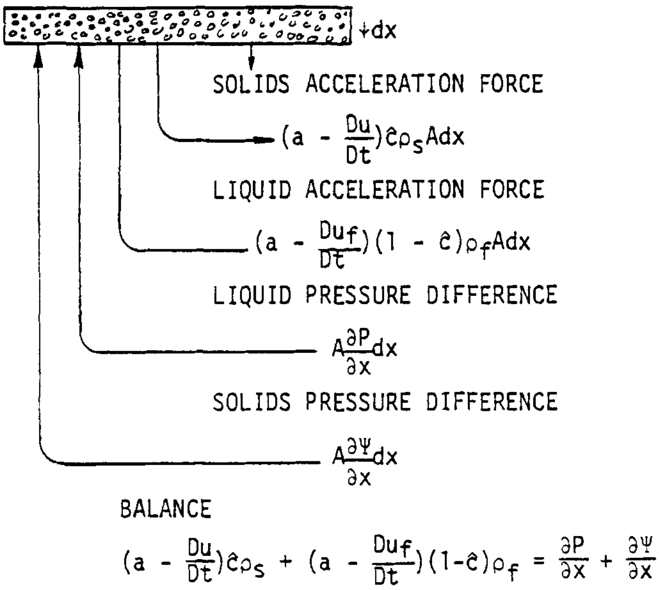

Figure 5. Thickening force balance.

In classical theoretical analyses, concentration C is taken as that of primary or unflocculated particles. But in clarification, a doublet, composed of two cohering primary particles, may not have a significant settling rate, and would not disappear from the supernatant in a batch test, within the times concerned (Fitch 1962). Floccules might have to grow to contain a great many primary particles before settling out. Willus made calculations assuming floccules grew to some specified size before disappearing. Therefore his C includes doublets, triplets, and n-tuples up to the specified size, as well as unflocculated primary particles. The C's thus calculated varied with t in a way consistent with observed behavior.

Smoluchowski theory also predicts values for the rate constant k. It has been elaborated and applied, largely by Camp. Fluid shear, however, in addition to causing collisions and flocculation, also affects the density of growing floccules, and limits the size they can reach. These aspects of flocculation kinetics will not be further discussed here. Pertinent references include: Fair and Gemmell (1964), Thomas (1964), Hudson (1965), Camp (1968), Lagvankar and Gemmell (1968), Parker, Kaufman, and Jenkins (1972), and Gemmell (1972).

THICKENING

Nearly all theories of thickening commonly assume, often tacitly, that all particles in a given neighborhood will settle at the same rate. They differ in what other assumptions are made, and/or in mathematical development. Different assumptions correspond to different physical models of the thickening process. Most developments in this direction involve taking more and more effects into account, and thus making fewer assumptions. Some of the mathematical developments have been most significant, in that they discover previously unrecognized physical relationships. Others differ superficially from preceding theories, in that they comprise only different (but equivalent) ways of expressing the same relationships.

To display the similarities and differences between the various thickening theories, a quite general differential equation will be developed. To obtain solutions of the equation, various terms will be neglected, and/or various assumptions will be made about the relationship between variables. This gives rise to the different physical models of thickening.

The general equation is derived by force balance across a lamina of suspension of thickness dx and area A (Figure 5).

The mass of solids in the lamina

$$= \hat{c}\,\rho_s A dx$$

The mass of fluid in the lamina

$$= (1 - \hat{c})\rho_f A dx$$

Acceleration forces on solids

$$= \left(a - \frac{Du}{Dt}\right)\hat{c}\,\rho_s A dx$$

Acceleration forces on fluid

$$= \left(a - \frac{Du_f}{Dt}\right)(1 - \hat{c})\rho_f A dx$$

Differential fluid pressure across lamina (assuming point contacts between particles)

$$= -AdP = \frac{\partial P}{\partial x} AdX$$

Differential solids pressure force across lamina

$$= -Ad\psi = \frac{d\psi}{dx} Adx$$

Summing forces

$$\left(a - \frac{Du}{Dt}\right) c\rho_s Adx + \left(a - \frac{Du_f}{Dt}\right)(1 - \hat{c})\rho_f Adx$$
$$= \frac{\partial P}{\partial x} Adx + \frac{d\psi}{d\psi} Adx \qquad (7)$$

In nearly all mathematical treatments, local acceleration terms are neglected and dropped. If they are to be dropped ultimately, it simplifies calculation to drop them here. As will be shown later, they are not negligible in all cases, which is the reason for including them in Equation (7). But if they are dropped, Equation (7) reduces to:

$$a(\rho_s - \rho_f)\hat{c} = \left(\frac{\partial P}{\partial x} - a\rho_f\right) + \frac{\partial \psi}{\partial x} \qquad (7a)$$

But $a\rho_f$ is the static head gradient. The term $(\partial P/\partial x - a\rho_f)$ is thus the dynamic pressure gradient $\partial \overline{P}/\partial x$. And the dynamic pressure gradient is that available to produce fluid flow through the porous medium represented by the solids structure. Also, in gravity settling $a = g$. So:

$$g(\rho_s - \rho_f)\hat{c} = \frac{\partial \overline{P}}{\partial x} + \frac{\partial \psi}{\partial x} \qquad (8)$$

Equation (8) differs from the one derived earlier (Fitch 1975) in that solids pressure ψ is defined in a slightly different way. In the earlier paper, ψ was defined as the solids stress transmitted mechanically through particle to particle contacts. However, as Dixon (1978) notes, stress can be transmitted from one particle to another hydrodynamically, if they are approaching each other. Pressure is transmitted through local zones of augmented fluid pressure, resulting from reluctance of the fluid to flow out from between approaching particles. Solids pressure ψ in Equation (8) is defined to include this hydrodynamic contribution. Thus total solids stress $\psi = \psi_s + \psi_h$.

Equations including local acceleration terms will be treated for certain special cases later.

Classical zone settling theory starts with the assumptions that solids stress is absent, so $\partial\psi/dx$ is zero, and that local accelerations are completely negligible compared to g. This leaves:

$$g\hat{c}\,(\rho_s - \rho_f) = \frac{\partial \overline{P}}{\partial x} \tag{9}$$

Also dynamic pressure gradient is related to fluid flux by:

$$\frac{\partial \overline{P}}{\partial x} = ku \tag{10}$$

The resistivity, k, is assumed in classical theory to be a function of $\hat{c}$. That is, $k = k(\hat{c})$, and the flow is Darcian, so:

$$u = \frac{g\hat{c}\,(\rho_s - \rho_f)}{k(\hat{c})} \tag{11}$$

or

$$u = u(\hat{c})$$

and since $C = \rho_s \hat{c}$

$$u = u(C) \tag{12}$$

This is the Coe and Clevenger model for zone settling.

COE AND CLEVENGER MODEL, $u = u(C)$

Continuous Thickening

Coe and Clevenger (1916) conceived the idea that there would be a critical concentration at which flux of solids to the underflow of a continuous thickener would be minimum. This "critical flux" would be the maximum that could settle into the underflow of a thickener at steady state. The critical concentration was not necessarily that of the feed suspension, as tacitly assumed earlier (Mischler 1912).

Coe and Clevenger deduced the thickener solids flux, corresponding to any zone concentration C and underflow concentration C_u, essentially by the method of Mischler. A much simpler derivation is due to Yoshioka et al. (1957).

The velocity of settling solids past a reference level has two components: the settling rate u of the solids with respect to the sedimenting suspension, and the velocity v of the suspension downward past the reference level. The total downward flux of solids is then

$$G = C(u + v) \tag{13}$$

By solids material balance, noting that $v = Q_u/A$, Yoshioka et al. write:

$$G = C(u + Q_u/A) = C_u Q_u/A \tag{14}$$

All that remained to do (Fitch 1962) was eliminate the term Q_u/A from double Equation (14). The result:

$$G = \frac{u}{\dfrac{1}{C} - \dfrac{1}{C_u}} \tag{15}$$

is equivalent to Coe and Clevenger's original equation.

Mining engineers worked in terms of "dilution" D rather than concentration C. Dilution was usually defined as tons of water per ton solids. D is related to C as follows:

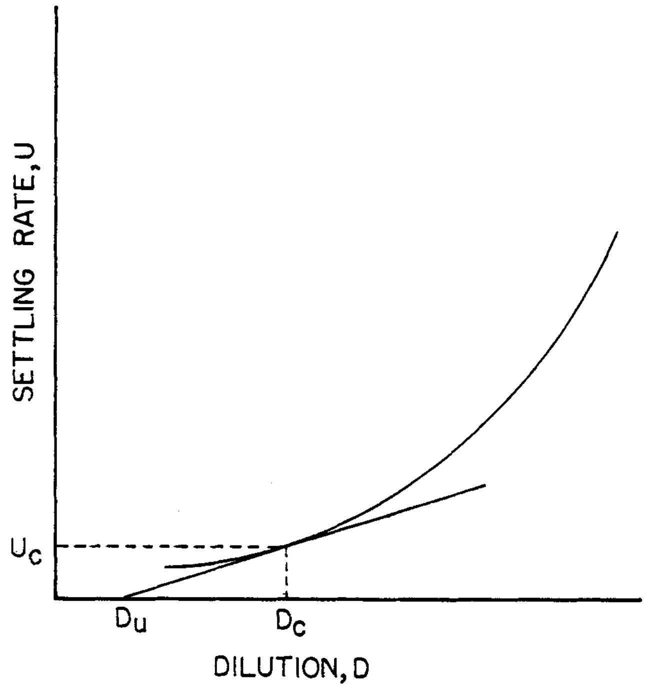

Figure 6. Mischler construction for critical flux.

$$\frac{1}{C} = \frac{D}{\rho_f} + \frac{1}{\rho_s}$$

Thus Coe and Clevenger expressed (15) as:

$$G = \frac{\rho_f u}{D - D_u} \tag{16}$$

The classical Coe and Clevenger method for finding the minimum G was to run batch settling tests at a series of initial concentrations, plug the results into Equation (16), and choose the lowest value of G as the critical or limiting value. There are, however, several graphical procedures for working up the settling data.

In the oldest (Mischler 1917) u is plotted against D (Figure 6). A line drawn through any point (D, u) on the curve, and through the point $(D_u, 0)$ on the D axis, will have a slope equal to $u/(D - D_u)$. A line drawn through $(D_u, 0)$ tangent to the underside of the curve will have the lowest possible slope, and hence value of $u/(D - D_u)$. The point of tangency therefore gives the

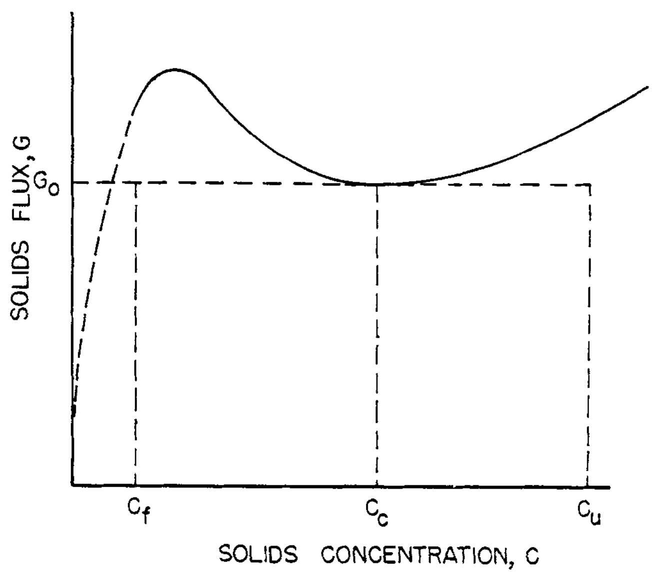

Figure 7. Thickener flux plot.

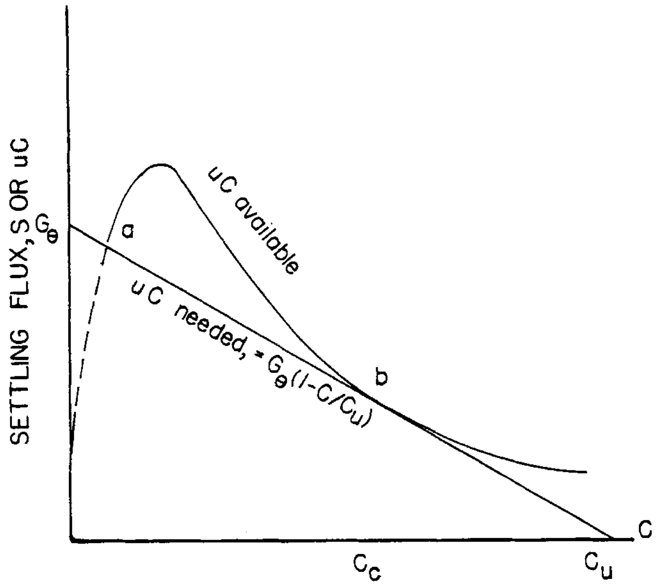

Figure 8. Yoskioka construction on Kynch plot.

dilution and settling rate for the critical zone. These values can be plugged into Equation (16) to give directly the critical or limiting G.

Alternatively, G may be calculated for each C and some assumed value of C_u (Mertes and Rhodes 1955, Yoshioka et al. 1957, Hassett 1958). A plot of G vs. C will appear as shown in Figure 7. The important empirical fact shown here is that there is (usually) a minimum in the plot of G vs. C. If, as is most frequently the case, the feed concentration is lower than that of this minimum, while the underflow concentration is higher, then the maximum solids flux that can pass to the underflow will be this minimum. On this plot, the critical concentration is the one showing the lowest value of G between feed and underflow concentrations.

The disadvantage of this plot, as compared to that of Mischler (Figure 6), is that a new curve has to be drawn for every value of C_u considered. Yoshioka et al. went on to derive a much more convenient construction that can be considered as a mapping of the Mischler construction onto a flux-concentration plane. Settling flux S or uC is plotted against C (Kynch plot). This represents the settling flux provided by the system. That needed, from Equation (13) is $G_\theta - vC$. And since by material balance $G_\theta = vC_u$

$$uC \text{ needed} = G_\theta(1 - C/C_u) \tag{17}$$

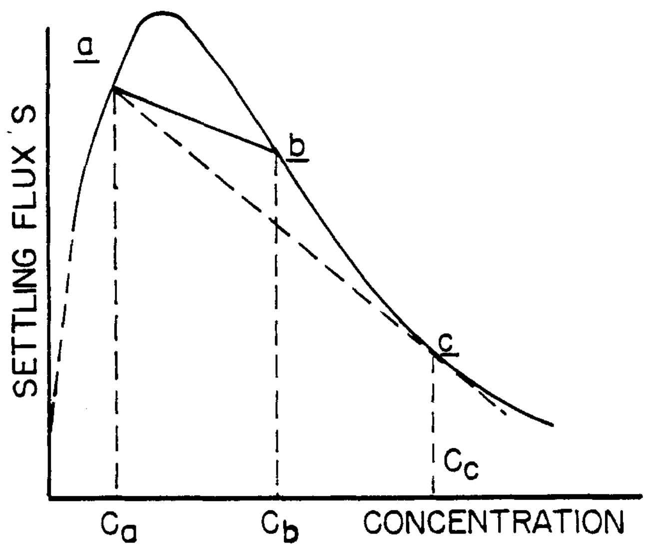

Figure 10. Discontinuity representation on Kynch plot.

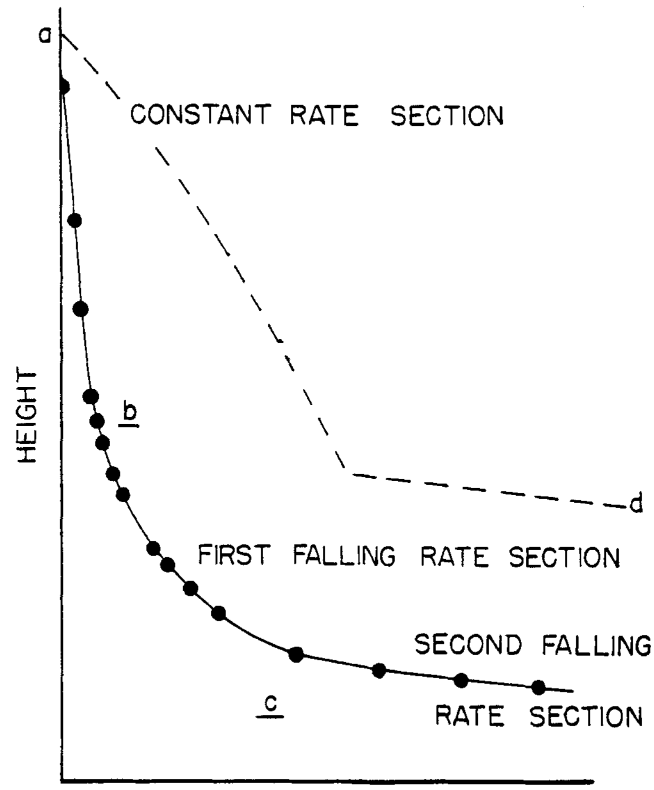

Figure 9. Batch settling curves.

Equation (17) defines a straight *operating line* on the diagram, having values of G_θ and C_u as parameters. The value of G_θ is given by its uC axis intercept, and that of C_u by its C-axis intercept. Such an operating line is shown in Figure 8.

The only concentrations that can exist occur where the the settling flux provided by the system equals that demanded by material balance. An operating line drawn through C_u on the C axis, and tangent to the underside of the uC curve, will locate a critical concentration corresponding to C_c in Figure 7. And the uC intercept of the operating line will measure the critical thickener flux G_θ.

Note that the constructions of Figures 6-8 are alternative mathematical procedures for determining G_θ. They do not add anything of physical significance to the theory. The Yoshioka construction of Figure 8, however, does display relationships in a particularly illuminating way, and is much used.

Batch Thickening

In batch thickening, a column of pulp (initially at uniform concentration in the line-settling or thickening regime), is allowed to settle. The level H of the sharp, pulp-supernatant interface is measured as a function of time. Ideally, either one or two subsidence-rate discontinuities occur (Figure 9). Two discontinuities divide the curve into a constant-rate section from a to b, a first-falling-rate section from b to c, and a second falling-rate section beyond c. There will sometimes be an initial increasing-rate section before constant rate to be reached, (curve a-d). This will be discussed later.

The constant-rate section corresponds to pulp settling at its initial concentration. From their constant-rate sec-

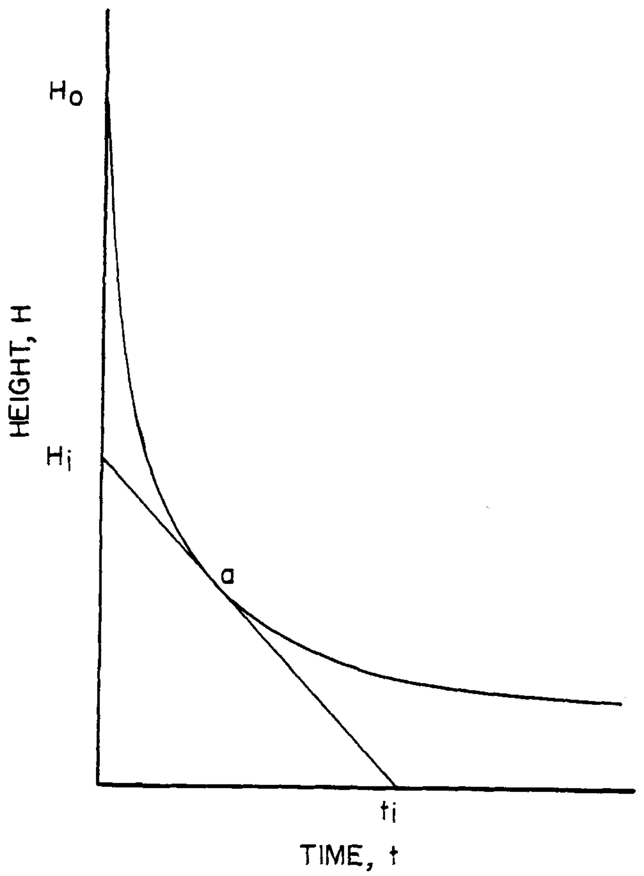

Figure 11. Kynch construction for concentration of pulp just below interface.

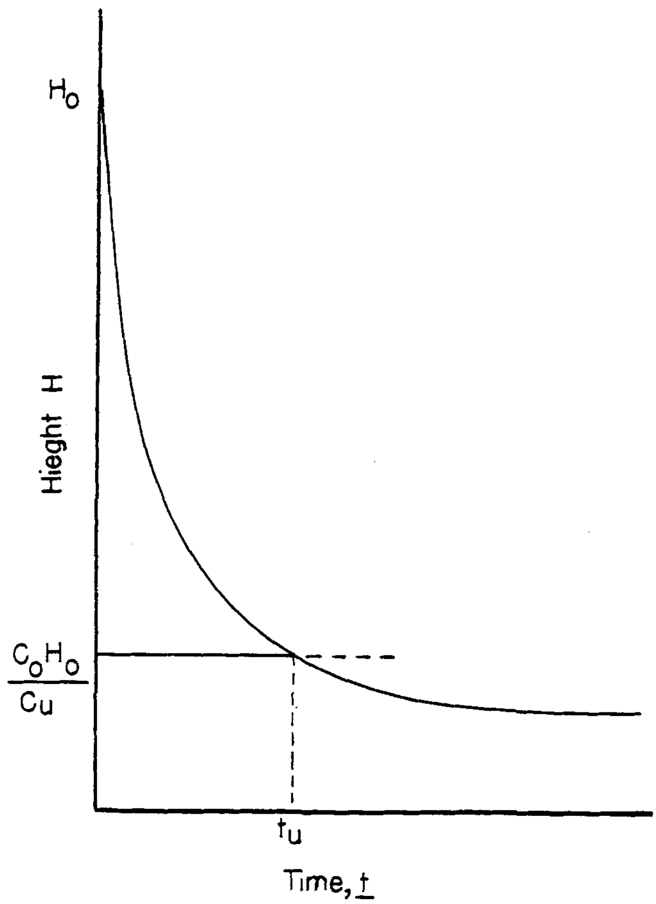

Figure 12. Talmage and Fitch relationship, (no compression).

tions, batch tests done at a series of initial concentrations give the data for developing Figures 6-8 and constitute the classical Coe and Clevenger method for determining the critical zone and flux.

The first-falling-rate, or transition section, arises from upward propagation of zones with ever higher concentrations but lower solids settling flux or capacity. The upward propagation of such zones was explained qualitatively by Coe and Clevenger (1916), and certain of the relationships inherent in later models were noted empirically by Work and Kohler (1940). But a full theoretical explanation with mathematical modelling was first given by Kynch (1952). By simple material balance or continuity, Kynch deduced that a concentration discontinuity should propagate in the direction of settling with a velocity α such that:

$$\alpha = \frac{\Delta G}{\Delta C} \tag{18}$$

On a flux plot (Figure 10), α is the slope of a chord drawn between points a and b on the curve, representing conditions directly above and below the discontinuity.

The locus of a zone of constant concentration C should propagate in the direction of settling with a velocity β such that:

$$\beta = \left.\frac{dG}{dC}\right|_C \tag{19}$$

Thus β for a given C is given by the slope of the tangent to the flux curve at C. This can be inferred from Equation (18) as a limiting case, when ΔC is allowed to approach zero. *It can also be derived, somewhat more elegantly,* from a one-dimensional continuity equation as done by Kynch.

Kynch's most important contribution, however, was to show how the solids concentration existing just below the pulp-supernatant interface at any point in a batch settling test, could be deduced from the shape of its settling curve. The mathematical relationships are best displayed by constructions on such a settling curve (Figure 11). A tangent drawn to the settling curve at any point a will intercept the H axis at some value H_i. It will intercept the time axis at some value t_i. What Kynch deduced was that, under the assumptions of the model, the solids concentration just under the interface at point a would be:

$$C_a = \frac{C_0H_0}{H_i} \tag{20}$$

The settling rate of solids just below the interface is the subsidence rate of the interface. Therefore

$$u_a = -\left. dH/dt\right|_a = H_i/t_i$$

From this it follows directly that

$$u_aC_a = \frac{C_0H_0}{t_i} \tag{21}$$

Equations (20) and (21) give the settling flux for every concentration that propagates to the surface in the batch test. From these, a settling flux (Kynch) curve such as shown in Figure 8 can be constructed, and from it, the flux capacity of a steady-state thickener can be determined. Thus, it was hoped, thickener area could be specified *from a single batch-settling test, rather than from* a battery of them as in the classical Coe and Clevenger method.

If initial test concentration C_0 lies in a range where the flux curve is concave downward ($d(uC)/dC$ negative) as at a in Figure 10, then those concentrations lying between points a and c would not propagate to the surface in the batch test. Discontinuity between concentrations a and c would propagate faster and overrun any that might form between concentration a and any intermediate concentration such as b. But this is no great loss. Comparing Figures 10 and 8 will reveal that, if the initial batch test concentration C_0 is at least as great as that of thickener feed, none of the missing concentrations in a batch test can be the critical one for continuous operation.

Talmage and Fitch (1955) simplified the method further. They showed how to determine the critical zone concentration and thickener flux directly from the batch settling curve, without going through the intermediate steps of first plotting a Kynch (uC vs. C) or Mischler (u vs. D) curve by means of Kynch constructions on the batch curve, followed by the appropriate (Yoshioka or Mischler) steady-state construction on the new plot. A horizontal line is drawn on a batch settling curve at an H value equal to C_0H_0/C_u (Figure 12). It intersects the settling curve at some time value t_u. If there were no compression zone, so that everywhere $u = u(C)$ (seldom true in practical cases), then the critical thickener solids flux would be

$$G = \frac{C_0H_0}{t_u} \tag{22}$$

Figure 6, the Mischler construction, Figure 8, the Yoshioka construction, and Figure 12, the Talmage and Fitch construction, are simply mappings of the same mathematical relationships onto three different coordinate planes. Kynch's third theorem can be construed as a transformation relationship for mapping from an $H - t$ plane to a $CdH/dt - C$ plane. It is analogous to the rules for transforming between McCabe-Thiele and Ponchon-Savarit planes and constructions in fractional distillation (Fitch 1951). The transformational aspects of Kynch theory have been discussed by Wallis (1963) and Jernqvist (1965).

Coe and Clevenger's zone-settling model, and the applications of Kynch theory to it, came to be known as the "flux theory." It was developed and elaborated by many investigators including Yoshioka et al. (1955, 1957), Hassett (1958), Tory (1961), Shannon, Stroupe, and Tory (1963), Wallis (1963), Moncrieff (1964), Shannon and Tory (1965). The Yoshioka papers, published in Japanese, were originally not known to most investigators. Thus Hassett, in a development that was remarkably comprehensive for its time, independently duplicated many of Yoshioka's ideas. Hassett also pointed out that any concentration existing at steady-state must be propagating upward (with respect to the pulp) with a velocity equal to the downward velocity of the pulp in the thickener resulting from underflow withdrawal—an approach that was extensively elaborated by Tory, and Shannon and Tory. Wallis further systematized Kynch's ideas on propagation of concentration zones and shocks, and generalized the Yoshioka construction of Figure 8. (His "characteristic velocity" is solids volume flux, $u\hat{c}$, which differs from Kynch's solids mass flux, uC, only by the constant factor ρ_s.) Montcrieff presents flux theory in terms of the Mischler construction (Figure 6), which also was relatively unknown at the time except as industrial know-how.

It is a defining characteristic of flux theory that $u = u(C)$. And it is a tacit assumption in all treatments cited above that flow is one-dimensional.

Numerous empirical equations have been proffered for the functional relationship between concentration C, and settling rate u. Also a "tube-and-tubule" model for the porosity of settling pulps, and hence for settling velocity, has been proposed by Gaudin and Fuerstenau (1960). Scott (1966) proposes an interfloc-intrafloc porosity adaptation of it. All these appear to have limited validity, and although significant for research purposes, are only secondarily relevant to the problem of thickener design.

One-dimensional flow would require that solids rain uniformly down from the level of feed introduction, onto the top of any subjacent zone of higher concentration. Over this distance, they would have some "upper conjugate concentration ," corresponding to point a in Figures 7 and 8. Blind adherence to the one-dimensional model led many authors to predict that such an upper conjugate concentration zone must exist in thickeners. It was well known in industrial practice that they did not, at least in general. The two-dimensional plunging of feed to its level of hydrostatic equilibrium (Figure 2) is now well documented. It does away with the theoretical need for an upper conjugate concentration. Blind adherence to the one-dimensional model would also lead to the conclusion that no pulp of any feed concentration other than C_a or C_b could exist in a thickener, since at such a C_f the operating and flux curves do not coincide.

Two-Dimensional Model

Continuous thickening is not overall one-dimensional, even as an approximation. Tarrer et al. (1974) present an alternative "free-settling" or $u = u(C)$ model that is completely two-dimensional. Cylindrical elements of thickening pulp, of an adjusted initial height, are translated across a basin as in Figure 3. Pulp at the bottom of the element is progressively withdrawn as it reaches underflow concentration. In this way, mass flow due to continuous withdrawal of underflow is superimposed on that due to settling. Mathematics calculate the relation between interface height and the concentration existing just below the pulp-supernatant interface, at any point along the path of the element across the basin. Dixon, Buchanan, and Souter (1974) correctly point out that the Tarrer equations violate one-dimensional continuity. The two-dimensional nature of the model is not made explicit in the original paper, but is deducable from the mathematical development.

Note that Tarrer's model makes no pretense of being valid where compression occurs and $u \neq u(C)$. It seems probable that real thickening will be more nearly correctly represented by a modified one-dimensional model than by a completely two-dimensional one.

DIXON MODEL, $u = u(C, Du/Dt)$

When inertial effects are taken into account while solids stress is still assumed absent, Equation (7) reduces to

$$g(\rho_s - \rho_f)\,\hat{c} = ku + \hat{c}\,\rho_s\frac{Du}{Dt} + (1 - \hat{c})\,\rho_f\frac{Du_f}{Dt} \tag{23}$$

As noted earlier, settling velocities are very low in thickening systems. So except at true or near-discontinuities, it might be anticipated that the local acceleration terms would be miniscule compared to fluid drag forces ku, and zone-settling behavior would be closely approximated. Furthermore, in a particulate system a true discontinuity could not exist. To settle from one concentration to another, particles have to come progressively closer over a finite distance. Over this distance a finite dC/dx exists, through which Du/Dt and Du_f/Dt would be finite and putatively small. There are, however problems, recognized by Hassett (1965) who wrote:

> "The mechanism of the change from a dilute concentration to its conjugate mud concentration is far from evident. The individual particles decelerate from u^r, their settling velocity in the disperse phase, to u, their slower settling rate in the high concentration phase. The only way they can decelerate is to find themselves in a higher concentration of solids, but it is logical to expect that they would overshoot and always be at above equilibrium concentration since the braking action would presumably only arise as a function of the over-concentration during the whole deceleration period."

Continuity consideration then led him to conclude:

> "This means that the particles seemingly decelerate in advance of their concentration increase."

Dixon (1977*a*, 1977*b*) made an extensive mathematical analysis of this model. He concluded: a) "There is no solids flux limitation associated with the free-settling part of the thickening zone in a continuous thickener," and b), "Concentration gradients in batch thickening can only develop in the compression zone."

These papers are disturbing. Their conclusions are forcefully argued, and strike at the very roots of both accepted theory and practice. And there is at least some experimental evidence that is consistent with the conclusions. In recent years, it has become evident that flocculent solids exhibit a compressive yield value over at least some part of what was formerly construed to be a free-settling range. And the flux-limiting concentration has been found often to occur in the compression range. Thus Dixon's conclusion might not entail much of practical consequence in the design of thickeners. Flocculent solids may, indeed, exhibit at least some compressive yield value over the entire range of concentrations observable in either the gradients that do develop in batch settling, or in the critical zones that are observed in continuous thickening. But then the question arises: Does settling behavior approach that predicted by existing models in the limit, as inertial forces become insignificant compared to those generated by gravity or fluid drag? In particular, if yield values and particle accelerations are very low compared to drag forces, would Kynch's theories relating to the propagation of concentration continuity waves be approximately valid, or would they fail completely? Dixon's conclusions appear to imply the latter.

Another question is this: Does the mathematical model represented by Equation (23) include everything that happens in a non-compression pulp? Is the model valid? And if not, what behavior would a corrected "free-settling" model predict?

Dixon, in a current paper (1978), proposes that there is indeed an effect operative in "free-settling" that is not included in the model. He proposes that there is a dynamic particle stress induced when particles approach one another, thus when $\partial C/\partial t$ is positive.

Dixon's theories are important, but are new and probably not well-known. Here, I present an overview together with a preliminary view of certain problems or anomalies they appear to entail.

Dixon argues that, for a discontinuity to exist, the pulp below must have at least sufficient rigidity to withstand the impulsive force generated by the change in particle settling rate. Since it is the solids structure that must withstand the impulsive force, it must be in compression.

He continues: a) An initial discontinuity will not propagate to a graded zone, and b) even if it did, it would lead by Kynch theory, under certain conditions, to a graded concentration zone with a discontinuity above and below. The upper of these discontinuities would have free-settling concentrations on each side, which violates his conclusions based on the argument in the preceding paragraph. "Thus even if it is argued that the initial discontinuity is unstable, so that after spreading a small amount its behavior can be calculated by assuming a balance between gravity and drag, making $u = u(C)$ (approximately) correct, the analysis can still lead to an inconsistency in the form of a stable discontinuity with free-settling concentrations on each side."

These arguments are based on behavior at a true or mathematical discontinuity. As noted above, and as recognized by Dixon, such a concentration discontinuity cannot exist in a particulate system. There must instead be a concentration front. If particle decelerations across the front are large with respect to g, settling rates may be augmented sufficiently to drive particles through any concentration loci that would be critical or limiting in the absence of inertial effects. On the other hand, if particles decelerations across the front were small with respect to g, one might expect that, at some point in the front, particle momentum would be expended after which Kynch behavior would occur. However, substitution of concentration fronts for discontinuities does not resolve the difficulties.

First, Dixon, Souter, and Buchanan (1976) devised a computer model of particulate settling, incorporating a concentration front of the type indicated above. The model, with inertial terms included, showed an initial discontinuity propagating to a concentration front of apparently constant thickness, under conditions such that the ever-expanding front of Kynch theory is obtained when inertial terms are omitted. Because of computer time limitations, however, model solutions are obtained only for very low values of a dimensionless number, N, measuring the relative ratio of gravitational to inertial effects. It is basic to the arguments presented above that behavior, when inertial effects are small compared to g (high N), might be qualitatively different than that occurring when inertial effects are large compared to gravitational ones (low N). Therefore, the computer results may not support a general conclusion that expanding fronts never occur.

Kynch theory predicts discontinuities (which are a special case of steady-state fronts) in some cases, and expanding fronts in others. But Dixon's computer solution showed a steady-state front developing where Kynch theory would predict an expanding one. Unsteady-state fronts are mathematically complex, and are hard to analyze logically. Steady-state fronts are much simpler and easier to understand. Accordingly, we seek some insight into the model by considering the possibility and/or characteristics of steady-state solutions to Equation (23).

Dixon's model differs from Equation (23) in that he omits the liquid acceleration Du_f/Dt, and argues that it should not be included. Omission of this term greatly simplifies the mathematics. But since it is included in our model, I consider what effect the liquid acceleration would have.

In a steady-state operation with a front rising with respect to the suspension (such as exists in a thickener), there is a constant flux of solids, and a constant flux of liquid, both towards the underflow. The solids are moving

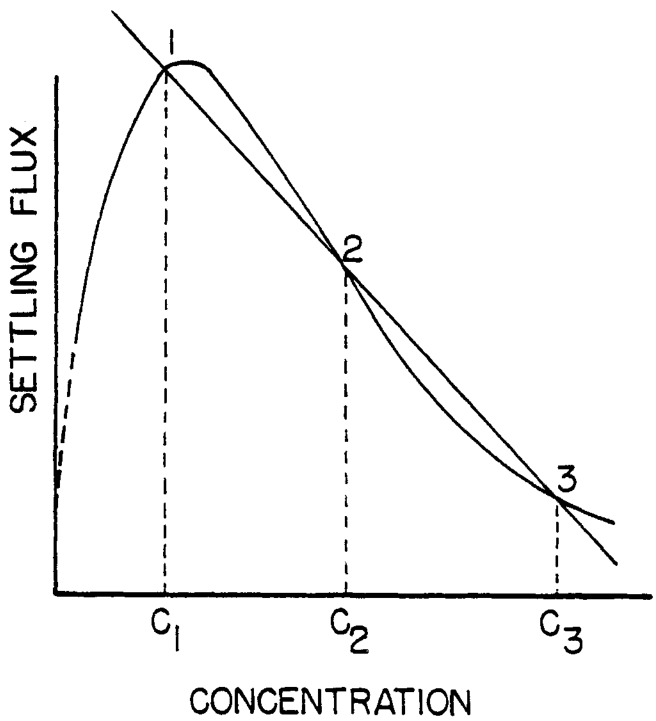

Figure 13. Constructions for steady-state discontinuities.

from lower to higher volume fraction, $\hat{c}$, and so must be decelerating. But the liquid is moving from where its volume fraction $(1 - \hat{c})$ is higher, to where it is lower. It must therefore be accelerating. According to our model and Equation (8), this would partially offset the effect of solids concentration. However, at least in most cases of practical interest, it appears that the solids deceleration would dominate, and both models would predict the same qualitative results.

An explicit equation for momentum balance through a steady-state front can be developed straightforwardly from Equation (23), together with the fact that $u\hat{c} + u_f(1 - \hat{c}) = 0$ (the total flux of the system, with respect to the system itself, is zero), that $(u - \beta)\hat{c}$ = a constant, and $(u - \beta)(1 - \hat{c})$ = another constant. (Constant solids and fluid fluxes past a steady-state front propagating with respect to the suspension at a velocity β.) But the resulting equation is sufficiently complicated to obscure its physical significance. In the arguments to follow, I postulate simply that where solids are decelerating, their settling rate is greater than it would be in the absence of the deceleration.

Fronts of constant thickness and shape will be steady-state referred to the locus of any given concentration in the front. Yoshioka or Wallis-type constructions on a flux plot should then correctly represent the mathematical relationships. The flux plot represents sedimentation in the absence of inertial effects, the operating line represents the continuity or material balance constraint. Figure 13 is such a plot; the operating line corresponds to "op line 2" of the Dixon paper (1977*b*).

As a first case, assume a thickener feed, or upper concentration, corresponding to point No. 1, and a critical zone or lower concentration corresponding to point No. 2. To comply with Dixon theory, assume, for this case, that pulp at C_2 has whatever compressive strength is needed to support a discontinuity between C_1 and C_2.

In a continuum model, there will be a complete, or mathematical, discontinuity between C_1 and C_2. Intermediate concentrations need satisfy no constraints, because they aren't there. But in a particulate model, concentration is required to pass through all intermediate concentrations, as they collapse from C_1 to C_2. And between C_1 and C_2, the operating line runs below the flux curve. The settling rate at any such concentration is less than would be produced by gravity alone. This means Du/Dt must be positive. To satisfy momentum balance the particles must be accelerating. But their concentration is increasing as they collapse. So to satisfy material balance they must also be decelerating!

Thus a steady-state or constant-thickness front between C_1 and C_2 is inconsistent with particulate theory.

As a second case, assume an upper concentration, now taken to be free-settling, corresponding to point No. 2, and a critical zone or lower concentration corresponding to point No. 3, also free-settling. A steady-state continuum solution for a concentration front is now possible. At any concentration between C_2 or C_3, the extra solids flux demanded by material balance is provided by augmentation of settling rates resulting from deceleration of the solids.

Since the operating and flux curves coincide at points No. 2 and No. 3, no augmentation of settling rates is indicated, so Du/Dt must become zero there. And $\partial c/\partial x$ must then also be zero. Therefore, the front approaches both C_2 and C_3 asymptotically. The mathematical solution shows a front extending to infinity both above and below.

The lower asymptotic tail can be cut off by allowing the concentration below the discontinuity to exhibit solids stress. Even a very small solids stress (either dynamic or static) would suffice, because most of the solids momentum will have been dissipated in augmenting settling rates (and overcoming correspondingly increased drag) through the rest of the front.

It seems reasonable to question the stability of this solution. A Dixon discontinuity with free-settling pulp above, and one with sufficient crush resistance to offset impulsive force below, should be stable. If gradients in the initial condition front are greater than those corresponding to the steady-state gradient solution, then the initial front might be expected to propagate to a stable Dixon discontinuity. If, on the other hand, the initial gradients are lower than in the steady-state solution, one might expect the initial front to expand, much as predicted by Kynch theory. But we do not yet know whether this would happen.

In particulate theory, the gradients through the initial front are geometrically limited. Depending upon how they compare with those in the steady-state gradient solution, it might be that the initial front could propagate either to a Kynch-like expanding front, or to a Dixon-front of approximately constant thickness.

Thus far, the theory presented by Dixon has been mostly negative in scope. It tells what does not happen. Or more precisely, it shows that certain previously expected results are inconsistent with the assumed mathematical model. But it also casts grave doubts on the validity of currently accepted theories and practices. What is needed is a reconstruction, a coherent picture of what does happen. And from an engineering standpoint, if Dixon's theories are valid, how do we go about designing thickeners?

It is hoped that Dixon will develop his ideas further. If not, it would seem a worthy project for some other mathematically-skilled investigator. And we cannot be satisfied with free-settling theory until it is done.

MICHAELS AND BOLGER MODEL $u = u(C, dC/dx)$

Michaels and Bolger (1962) assume that the solids structure in a compressing pulp will have a resistivity k, and a compressive yield value, σ, both of which are functions of solids concentration. They neglect both particle acceleration and dynamic solids stress. Under these assumptions, from Equation (8)

$$g\left(\frac{\rho_s - \rho_f}{\rho_s}\right) C = \frac{\partial \overline{P}}{\partial x} + \frac{\partial \psi}{\partial x}$$

Where pulp is actually compressing (and neglecting ψ_h), $\psi = \sigma$. In such a case, (8) can also be written:

$$g\left(\frac{\rho_s - \rho_f}{\rho_s}\right) C = ku + \frac{d\sigma}{dC}\frac{\partial C}{\partial x} \tag{24}$$

and $u = u(C, \partial C/\partial x)$.

Michaels and Bolger concerned themselves with the initial batch settling rate of pulp in the compression regime. Initially only pulp at the very bottom of the column is compressing, so $\psi = \sigma$ at the bottom of the column. Making a force balance on so'ids over the entire initial height H_0 of the non-compressing plug (and neglecting wall effects, which Michaels and Bolger do not):

$$\text{Net Gravity force} = g(\rho_s - \rho_f)\,\hat{c}\,H_0$$

$$\text{Fluid drag} = -\,ku_iH_0$$

$$\text{Solids stress} = -\,\sigma$$

So

$$g(\rho_s - \rho_f)\,\hat{c}\,H_0 - ku_iH_0 - \sigma = 0 \tag{25}$$

Yield height Y is now defined as the height of solids that can be sustained by a solids stress equal to σ at the bottom, where u is zero. Then:

$$g(\rho_s - \rho_f)\,\hat{c}\,Y = \sigma$$

And u_∞ is defined as settling rate in the absence of solids stress, or when H_0 is infinite, so:

$$g(\rho_s - \rho_f)\,\hat{c}\,H_0 = ku_\infty H_0$$

Substituting appropriately in (25), and simplifying, one obtains the Michaels and Bolger relation

$$\frac{u_i}{u_\infty} = 1 - \frac{Y}{H_0} \tag{26}$$

To discover values of parameters Y and u_∞, for a given concentration, make a series of settling tests at that initial concentration and various values of H_0. A plot of corresponding u_i vs. $1/H_0$ should be a straight line. Its intercept with the $1/H_0$ axis gives the value of $1/Y$. Its intercept with the u_i axis gives u_∞.

An equation equivalent to (26) was derived from the same physical model by Dick (1970). In Dick's form of the equation, yield height Y is implicit, but was not isolated and identified as a parameter. And he did not perceive the equivalent of Michaels and Bolger's method for evaluating the parameters from experimental data. Perhaps because appropriate parameters were not found, he concluded that the model was defective, and rejected it in favor of an empirical "retardation factor" equation presented earlier (Dick and Ewing 1967). His retardation factor equation can be shown equivalent to

$$\frac{u_i}{u_\infty} = \frac{1}{1 + H_D/H_0}$$

The "Dick height parameter," H_D, can be identified as that at which $u_i = u_\infty/2$. The physical significance of this parameter height is not clear, but the equation seems to fit activated sludge batch settling quite well. As shown by Dick, the two models differ significantly only when initial heights H_0 are small.

The Michaels and Bolger model was extended mathematically to regions of crushing or compression below the initial plug, and directly to the problem of thickener design, by Fitch (1966).

Where solids are actually consolidating, Equation (24) applies. Its unknown parameters are k and $d\sigma/dC$. Resistivity k can be expressed in terms of settling rate u_∞ in the absence of solids stress gradient, hence when either σ is zero (free settling) or $\partial C/\partial x$ is zero. That is;

$$k = g\left(\frac{\rho_s - \rho_f}{\rho_s}\right)\frac{C}{u_\infty}$$

Parameter $d\sigma/dC$ can be expressed in terms of the concentration gradient $(dC/dx)^*$ existing in a column after consolidation is complete and settling rate u is zero. Thus, from (24)

$$\frac{d\sigma}{dC} = g\,\frac{\left(\frac{\rho_s - \rho_f}{\rho_f}\right) C}{\left(\frac{dC}{dx}\right)^*}$$

Substituting in (24) and simplifying

$$\frac{dC}{dx} = \left(\frac{dC}{dx}\right)^*\left(1 - \frac{uC}{u_\infty C}\right) \tag{27}$$

In steady-state thickening, uC is the solids flux, G needed for continuity or material balance, and is that plotted by the operating line on a Yoshioka construction (Figure 8). $u_\infty C$ is the solids flux S in the absence of solids stress gradient. For any given C, u_∞ can be determined with theoretical precision from u_i vs. $1/H_i$ plots according to the method of Michaels and Bolger. Practically, it will be closely approximated by the initial settling rates in batch tests with large initial heights.

Further manipulations and substitutions yield

$$z = \int_{C=0}^{C_u}\left(\frac{S}{S-G}\right)\left(\frac{dx}{dC}\right)^* dC \tag{28}$$

where z is the depth of the compression zone in continuous thickening. A procedure for graphical integration of (28) is given in the cited paper.

Shirato et al. (1970) recognized similarities between the Michaels and Bolger model, and that of Terzaghi (1943) for soil consolidation. They take what is basically a Terzaghi approach, but modify and develop it to suit the self-consolidation occurring in thickening. Their objective is to explain batch consolidation, and following soil mechanics approach, their fundamental differential equation expresses $\partial\overline{P}/\partial t$. However, instead of using volume concentration $\hat{c}$ and sediment depth x as independent variables, they develop equations based on ω, the volume of solids per unit area above a given level, and void ratio, e, or volume of voids per volume of solids. Thus $\hat{c} = \partial\omega/\partial x$, and $e = (1 - \hat{c})/\hat{c}$. As a result, their

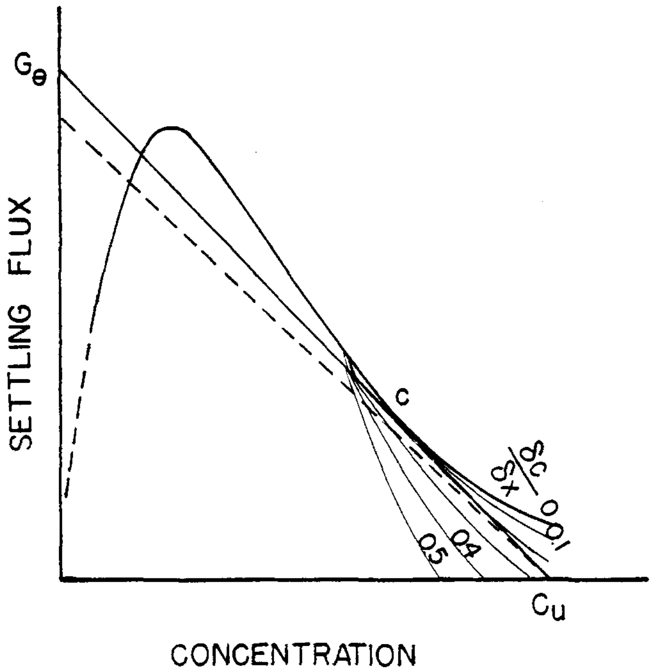

Figure 14. Flux plot with $\partial c/\partial x$ as parameter.

equations are different in form from ones that could be developed in $\hat{c}$, x, and t. They have the advantage of much simpler boundary conditions for the case of batch thickening.

To solve the equations there is needed an empirically-determined relationship between permeability and void ratio, also one between porosity (or e) and solids pressure ψ.

For the permeability relationship, Shirato et al. assume that the Kozeny equation will give the functional relationship, and its parameters are determined from the initial settling rates observed in batch settling. This latter step is not strictly valid theoretically, since the authors assert that in all tests, the original concentrations were in the compression regime. In such cases, from Equation (26), u_i should be a function of H_0. Their experimental results, however, show u_i apparently independent of H_0. So yield height Y must have been very much smaller than H_0 in all cases. Thus the error theoretically introduced would be undetectibly small. And, of course, if u_i had proven to be a function of H_0, the true u_x could have been discovered by the plotting method of Michaels and Bolger.

To determine compressibility they allow batch tests to continue until settlement is complete. The final height H_x, is plotted against ω_0, the volume of solids present, for a number of tests with different values of ω_0. Then $\hat{c} = d\omega_0/dH_x$. From this, their compressibility coefficients can be calculated.

Shirato et al. determined experimentally $\overline{P}$-profiles and $\hat{c}$-profiles at various times in batch settling tests, made on *zinc oxide and ferric oxide suspensions. In all cases,* experimental results coincided closely with those predicted by their theory.

They did not extend their theory to continuous sedimentation and the design of thickeners.

Gould (1974), after some surprising assertions about existing art, goes on to repeat the reasonings and equations given by Fitch (1966).

Shin and Dick (1974) made a force balance over a section of a settling column extending from the top surface of the supernatant, down to some level x in the sediment zone. Thus they derive an integrated form of Equation (8). They also note that $\partial\overline{P}/\partial x = uk$, but did not carry mathematical modeling beyond this: They made batch tests on a water-treatment sludge, obtaining concentration and $\overline{P}$ profiles much in the manner of Shirato et al. They observed that both solids stress and permeability at any given concentration were relatively unaffected by the position and time at which the concentration occurred. This supports the Michaels and Bolger model. (The Shin and Dick paper presents some interesting and theoretically provocative experimental data. Thus compression yield value σ is shown to vary, within experimental error, as the cube of shear yield value τ.)

Adorjan (1975) combines the equivalent of Equation (24) with continuity to develop the general partial differential equation for Michaels and Bolger thickening in C, t, x, coordinates. Like its Shirato equivalent in e, t, ω, coordinates, and as noted in the Fitch (1966) paper, "It is formidable." It is a commentary on the rapid growth of computer art since 1966 that both Shirato and Adorjan both immediately provide finite difference equivalents to be used for computer numerical solutions.

Adorjan expresses the compressibility relationship as

$$\sigma = a'\sinh(C - C_z),$$

and the permeability relationship as

$$1/k = bC^n$$

Using these functions, which he determined empirically, he goes on to provide representative solutions for batch thickening, and a simplified ordinary differential equation, (corresponding to Equation (27) of this paper), for steady-state thickening. Except for the empirical compressibility and permeability functions, Adorjan's paper does not present any basically new or different theory (Pearse 1977). But it provides a rather complete recapitulation and explication of what had been done before. Mathematical steps, some only implied or indicated in earlier papers, are carried out in detail, and methods for numerical solution of equations are provided.

INTERRELATIONSHIPS BETWEEN ZONE SETTLING AND COMPRESSION

Although there are academic attempts to explain all thickening phenomena on the basis of zone-settling and Kynch theory, it was never doubted in industrial practice that compression zones exist and have to be allowed for in the design of thickeners. It is, however, a tacit assumption of the original Coe and Clevenger model that critical zones would not occur in compression. It was thought that the channeling observed in compression and believed to be its concomitant would always augment settling rates sufficiently so that compression concentrations would not be flux-limiting. This, however, was later modified by the "three-foot rule":

> "A frequently used empirical design method is to determine a detention time for pulp in compaction according to the procedure recommended by Coe and Clevenger for metallurgical pulps and to specify that the (design) depth of the compaction zone shall not exceed three feet. For slowly compaction pulps this results in an area demand in excess of that required by zone settling." (Fitch 1966)

Thus, although unrecognized by many academic reviewers of the literature, thickeners were not being specified

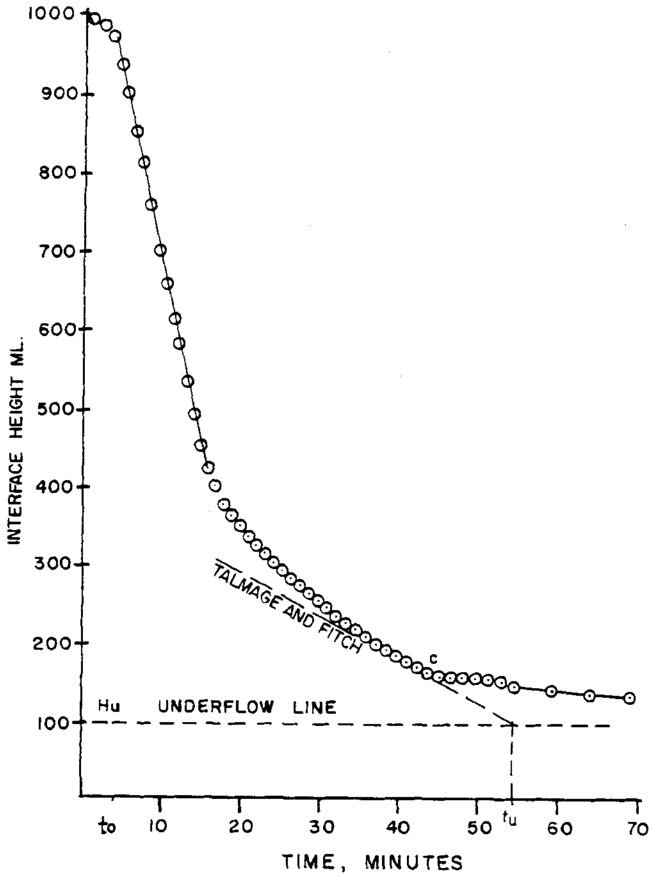

Fig. 15. Talmage and Fitch construction (with compression).

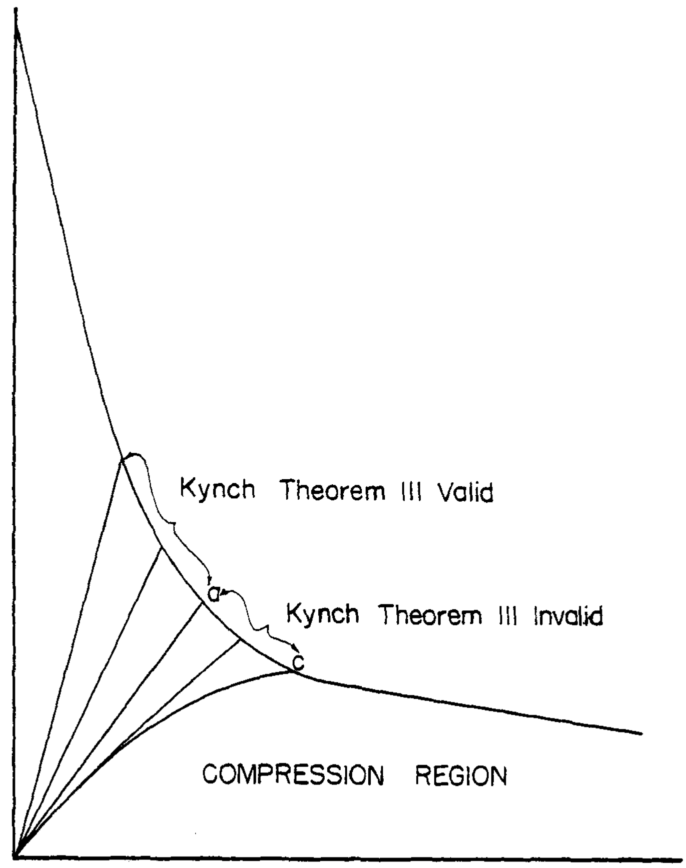

Figure 16. Region where Kynch construction is invalid.

in practice solely by Coe and Clevenger zone settling theory, or by any of the several graphical procedures based on it. And the same 1966 paper also showed how critical zones could form in the compression regime, based on the Michaels and Bolger model:

According to the model, $u = u(C, \partial C/\partial x)$, from which it follows that settling flux S ($= uC$) is $S(C, \partial C/\partial x)$. The functions can be represented graphically as a series of plots of S vs. C (Kynch plots) having different values of $\partial C/\partial x$ as parameter (Gould 1974). A hypothetical set of such curves is shown in Figure 14. The curve for $\partial C/\partial x = 0$ represents the settling flux where there is no solids stress gradient ($\partial \sigma/\partial x = 0$). This condition is satisfied by any pulp in zone settling. Therefore, the $\partial C/\partial x = 0$ curve in compression is simply an extension of the *free settling curve.*

According to the model [Equation (24)], the solids flux, therefore also settling rate, at any given concentration is maximum when $\partial C/\partial x$ is zero (reverse concentration, $\partial C/\partial x$ negative, being ruled out on physical grounds). This maximum settling rate, $u_{\max}$, will also be the rate approached in batch tests as H_0 becomes much greater than Y. And the curve for $u_{\max}C$ vs. C extends through both zone settling and compression regimes.

A Yoshioka construction on a plot of $u_{\max}C$ vs. C will establish a critical thickener flux G_θ. If the critical concentration, defined as that at which the operating line is tangent to the maximum flux curve, lies in the zone settling regime (Figure 8), the G_θ found will give the classical Coe and Clevenger zone settling unit area. If the *critical concentration* lies in the compression region (Figure 14) then the unit area thus determined will be a lower bound for that actually needed. The point of *tangency c* is a "pinch point." At such a point, $\partial C/\partial x$ is zero, and an infinite height is needed to get past it. Lowering the operating line (lower G_θ and/or lower C_u) causes it not to pass through a region where $\partial C/\partial x$ is zero (dotted line, Figure 14). The lower the operating line, the higher the value of $\partial C/\partial x$ at any given concentration in the thickener and therefore the lower the integrated depth of the compaction zone. But with easily compressible pulps (or if the pinch point lies in an easily compressible range) the clearance necessary between maximum flux and the operating line at the pinch point may not have to be at all large to yield a shallow compaction zone.

A consequence of the above is that the Coe and Clevenger settling procedure should, under certain conditions, come close to predicting unit area needed, even though the critical concentration lies in the compression regime. The conditions are that the initial heights H_0 in the batch settling tests be many times the yield height Y, so that the settling rates observed are close to $u_{\max}$ and that the pulp be easily compressible (or in an easily compressible range). This would account for the fact that zone settling procedures reputedly (Dick 1970) predict area requirements for final settling of activated sludge even though it was recognized (Dick and Ewing 1967) that the pulp was actually in compression over the concentration range in which the settling tests were run.

The above was probably the rationale implicit in Moncrieff's (1964) approach, using the equivalent Mischler construction.

In pulps more difficult to compact, the operating line may have to be lowered considerably to avoid too great a compression depth. As will be obvious, there is a tradeoff between high G_θ and/or C_u on the one hand, and low compression depths z on the other.

The very existence of a compaction zone invalidates (at least theoretically) all thickeners design methods based on Kynch theory (Fitch 1966). And in particular, it invalidates the Talmage and Fitch procedure.

In classical Coe and Clevenger theory, it is assumed that critical zones would not form in compression (because of channeling), and also that the "compression point" in a batch settling curve (Figure 9), marked the point at which the pulp just below the interface passed from zone settling to compression. By Kynch reasoning the surface pulp just before the compression point should be at the most limiting concentration existing in the free-settling regime. Therefore, the Talmage and Fitch procedure is to draw a construction line tangent to the low-concentration leg of the settling curve at the "compression point" (Figure 15), unless H_u is greater than H_c. The characteristic t_u to use in Equation (21) is then that at the intersection of the tangent and the H_u line.

Of course if the surface pulp just before the "compression point" is already in compression (which now seems likely in many cases) the basic postulate of Kynch theory, that $u = u(C)$, is not valid, so Kynch theory becomes unsound there. But even if the surface pulp is in zone settling, Kynch's "third theorem" constructions will not be valid, if there exists a zone of compressing pulp at the bottom of the column.

Kynch's "third theorem" construction, on which the Talmage and Fitch procedure is based, requires that the zone of constant concentration propagating to the interface originate at the bottom of the column at time zero, and propagate to the surface at a constant velocity, β. But, as shown in Figure 16, the interface between zone-settling pulp and compressing pulp in a batch test does not propagate linearly from the origin. It propagates with an ever-decreasing slope. There is some concentration C_a that at time zero propagates at the same velocity as the zone-settling compaction interface. For this concentration, as for all lower ones, Kynch theorem III holds. But higher concentration zones do not originate at time and height zero. They rise from the compaction interface, with concentrations such that their propagation velocity equals that of the compression interface at the point of their origins. Thus between points a and c, Kynch theorem III conditions are not satisfied, and so theorem III is not theoretically valid. In practice, however, the shape of the settling curve is such that the error introduced from this source might not be large.

Although Kynch theorem III is not theoretically valid when the pulp below the interface is in compression, its constructions still may serve to locate approximately the critical or limiting solids flux if the batch test is carried out in a sufficiently deep column.

As zones or loci of two different concentrations propagate upward, they in most cases diverge. The concentration gradient between them therefore decreases. As concentration gradient decreases, the influence of solids stress on settling velocity also decreases. When the concentration gradient becomes small enough, the solids in it settle at essentially their $u_{\max}$ and Kynch's assumption, namely that $u = u(c)$, becomes approximately valid. If the column is deep enough so that a zone of constant concentration propagates at approximately its Kynch rate over nearly its entire path, it will approach the surface at approximately the same time and height as it would had it propagated the entire distance at its Kynch rate. (That is, the percentage error may be small.)

However, $\partial C/\partial x$ does not continue to dwindle, and hence become arbitrarily small, over the entire column height. In order for a layer of pulp to exist in the compression regime (if not there initially) it must be subjected to a solids stress from overlaying solids. The pulp just below the pulp-supernatant interface has no overlaying solids. It cannot have a concentration higher than the most concentrated zone settling one. Therefore, strictly speaking, a Kynch construction, even on the settling curve from a very deep column, will not show either the $u_{\max}$ or the concentration of pulp just below the interface, when pulp just below the boundary layer is at compression. There is rather a zone, just below the surface, through which concentration changes rapidly to approach that in a subjacent region where $\partial C/\partial x$ becomes negligibly small. Thus loci of concentrations in the compression regime never reach the surface. However, the top zone of increasing concentration will be very shallow when the subjacent zone is at an easily compressible concentration. (It must become deeper as the subjacent concentration becomes greater.) It is the settling rate in this subjacent region, near but not at the interface, that will control the subsidence of the interface. And it is the concentration and settling rate of this subjacent layer that is found (approximately) by Kynch constructions on a deep column test curve.

The propagation paths for loci of constant concentration (Kynch paths) through compressing pulps are mathematically complicated. No solutions have, as yet, been obtained. Therefore we are not able to predict, from theory, by how much Kynch constructions will be in error, or how deep a "deep column" would have to be to permit reasonably close approximation of critical flux by Kynch or Kynch-related methods. But a current paper by Wilhelm and Naide (1979) shows experimentally that Talmage and Fitch constructions on settling curves from batch tests carried out in two-liter graduates (under certain conditions, and approximately) predicted the unit areas observed in continuous thickening.

The Wilhelm and Naide paper does not draw, at least explicitly, the above conclusion. Its theoretical presentation is based on a misconception of existing art, and may be misleading. It makes a considerable point of the fact that, under the assumption that Kynch constructions are valid down to H_u, the value of t_u is to be determined by the intersection of the H_u line with the settling curve (Figure 12), and not by its intersection with a tangent to the settling curve drawn at some hypothetical compression point above H_u. They imply that this represents a new model, different from that of the Talmage and Fitch method, and a "correct" analyses of batch settling curves. But the primary derivation and conclusion of the Talmage and Fitch paper is precisely the same! This has been reiterated and further explained in subsequent publications (Purchas 1977, Fitch 1975). When Kynch constructions are valid down to H_u, there can be no "compression point" above H_u. And when there is no compression point above H_u, t_u is to be read at H_u on the settling curve (Figure 12). Thus there is nothing new or different in the Wilhelm and Naide model. Its interpretation of batch settling curves is not theoretically "correct" except under the conditions or assumption that $u = u(C)$ (Kynch assumption). And as discussed, its interpretation, as of any Kynch model, is never completely "correct" theoretically where compression exists at all.

Wilhelm and Naide also present an indirect method for determining unit area. By means of Kynch constructions on the settling curve, they generate a plot of settling rate u (presumably $u_{\max}$) as a function of concentration. By curve fitting, they represent the function as piecewise a power function. Then, by an algebraic equivalent of Figure 8, they determine the critical flux or unit area. There is nothing theoretically wrong with this, if Kynch constructions can be assumed valid. But it is to bypass such rigamarole that the Talmage and Fitch construction was conceived. For any given underflow concentrations calculate H_u. Read corresponding t_u from the settling curve. Determine flux G directly [Equation (22)]. It gives the same answers as the roundabout procedure. Wilhelm and Naide demonstrate this for the special case that the settling rate is piecewise a power function of concentration. But it is necessarily true for any function. The two methods are mathematically equivalent. And the roundabout method neither adds anything to the validity of the results, nor subtracts anything from its theoretical limitations.

Notwithstanding its misconception of existing art, the Wilhelm and Naide paper is significant. It is the first to recognize and utilize the deep column principle for Kynch interpretation of a batch settling curve, and it is one of the few that compare experimentally the predictions of a batch settling model, with actual continuous thickening of the same material. Much more experimental work of this kind should be done.

The presence of compression invalidates also a procedure given by Michaels and Bolger for determining floccule size and internal solids concentration from settling data. They treat the entire floccule, including its internal or entrained water, as the settling entity. The volume fraction occupied by such floccules or entities in the suspension is $\hat{c}_e$, which is some factor j times that occupied by the ultimate solids phase particles alone. Then, by the empirical Richardson and Zaki (1954) relationship for low Reynolds number settling

$$u = u_0(1 - j\hat{c})^{4.65}$$

Or

$$u^{1/4.65} = u_0^{1/4.65}(1 - j\hat{c})$$

If (and only if) u_0 and j are constants, a plot of $u^{1/4.65}$ vs. $\hat{c}$ should be linear. The intercept of the plotted line with the $\hat{c}$ axis then measures the internal floccule concentration $\hat{c}_f$. Its intercept with the $u^{1/4.65}$ axis gives the value of the parameter $u_0^{1/4.65}$. But if the individual floccules are shrinking and/or splitting, neither u_0 or j remain constant. The plotted curve will not be linear. Further, its slope at any point does not then measure the effect of $\hat{c}$ alone, as would be required for validity of the Michaels and Bolger construction. A tangent to the curve would not necessarily intercept the $\hat{c}$ axis at $\hat{c}_f$. Use of such a tangent construction to measure shrinking and/or splitting of floccules during thickening (Javaheri and Dick 1969) therefore depends for validity upon the absence of the very effects it purports to measure. It is theoretically unsound.

Mathematically expressed, Michaels and Bolger show

$$\left[\frac{\partial(u^{1/4.65})}{\partial\hat{c}}\right]_{j,u_0} = ju_0^{1/4.65}$$

But, the slope of the curve is $d(u^{1/4.65})/d\hat{c}$, and

$$\frac{d(u^{1/4.65})}{d\hat{c}} = \frac{\partial(u^{1/4.65})}{\partial\hat{c}} + \frac{\partial(u^{1/4.65})}{\partial j}\frac{dj}{d\hat{c}} + \frac{\partial(u^{1/4.65})}{\partial(u_0^{1/4.65})}\frac{d(u_0^{1/4.65})}{d\hat{c}}$$

KOS MODELS

The Michaels and Bolger model had been tested experimentally by Shirato et al. (1970) and by Shin and Dick (1974). Both obtained results consistent with the Michaels and Bolger model. However, Kos obtained results from continuous tests that were not. (Kos and Adrian 1974, Kos 1978). And even before Kos' results it was recognized (Fitch 1966, 1972, 1975) that the model was inadequate to explain observed continuous thickening results such as obtained by Scott (1970). Hence, it appeared inadequate to predict continuous thickening behavior.

Force balance and continuity equations for Kos' models are the same as for the Michaels and Bolger model. Thus his basic differential equations are equivalent. He derives them, however, in yet another coordinate system, a material one referred to the motion of particles. This gives them in a form he finds convenient for batch test calculations. But by transformation to spacial coordinates he obtains the equivalent of Equation (8). He gives, particularly in his thesis (1978), a very complete mathematical development of the basic differential equations.

The physical novelty of Kos' models lies in the functionalities of the constitutive relations for compressibility σ and resistivity k. Michaels and Bolger assume $\sigma = \sigma(C)$ and $k = k(C)$. In an approximate model Kos assumes $\sigma = \sigma(C)$ but $k = k(C, \partial\bar{P}/\partial x)$. In the more exact model he assumes σ also is $\sigma(C, \partial P/\partial x)$.

Kos advances a plausible hypothesis for the dependence of k upon $\partial P/\partial x$. He assumes that liquid passing the solids does so mostly in channels. Floc forming the walls of these channels has some shear yield value τ, which is a function of the floc concentration. When the hydraulic wall stress caused by flow of liquid through the channel exceeds τ, the walls erode away. The channels flush out to a larger diameter. Thus channel diameter, and hence the permeability of the floc structure, always adjust to maintain the hydraulic stress equal and opposite to the solids yield value τ at the channel walls. And in a secondary way, compressive yield stress σ must be weakly related functionally to permeability.

For Kos' approximate model, Equation (8) can be expressed functionally as

$$f_1(C) = f_2(C, \partial C/\partial x, u) + f_3(C, \partial C/\partial x)$$

So for this model

$$u = u(C, \partial C/\partial x)$$

Behavior is thus a function of the same kinematic variables as in the Michaels and Bolger model. It can be handled mathematically in much the same way. The difference is that the index of compressibility, $d\sigma/dC$, is not independent of u. The values obtained after settling is complete ($u = 0$) are not valid where $u \neq 0$, as they are in the Michaels and Bolger model.

The functional relationship between u, C, and $\partial C/\partial x$ would be rather difficult to map from batch settling tests, but should be easily derivable from concentration profiles in a series of batch upflow or semi-fluidization tests (Fitch 1975). Upflow tests had been suggested earlier by Larsen (1968) and Busch (1970), but would have been unnecessary had the Michaels and Bolger model been valid.

An aspect of Kos' theory not reflected in his mathematical models, is that he considers channeling a characteristic and concomitant of compression. As soon as the floc structure has sufficient concentration to support any solids stress, it channels. This goes beyond traditional theory, which might be summarized as follows

"Associated with compression is a breaking up of the pulp and a channeling or short-circuiting of water upward through it. Presumably it takes a certain coherence or yield value in the pulp to support such channeling." (Fitch 1962)

The Kos model derives from a substantial but unique test program on continuous thickening. And, it resolves some difficulties encountered in trying to explain some less complete data and results from a very few other continuous thickening tests. But the model has not yet been widely tested or appraised. Its significance is that it seems reasonably consistent, so far, with the limited amount of experimental evidence available, and replaces a theory (Michaels and Bolger) that has been found wanting. It opens a new door to our thinking, as did the models of Kynch and of Michaels and Bolger before it, and is the only basically new rational approach since that of Michaels and Bolger.

It is too early to predict how generally valid and useful the Kos model will be. As will be evident from the developments of this paper, it is the least restricted and most general of those proposed. Both the Michaels and Bolger model, and that of Kynch, can be comprised mathematically as special cases of it. We know that it cannot be a complete and exact general model, however, because it turns out that the settling behavior of thickening pulps can be a function also of both the solids concentration at which the floc structure is originally formed, and of its shear history.

FLOC STRUCTURE

Floc structure in a sedimenting suspension appears to be complex, fragile, and mutable. This has been long recognized in clarification. There is an extensive literature on how flocculation and coagulation can be improved or modified in this regime.

For many years little attention was paid to floc structure in the thickening art. In zone settling, it was assumed that $u = u(C)$, which tacitly entailed that the floc structure be fixed at any given concentration. And for metallurgical pulps this may be approximately valid. In compression it was assumed that the floc structure, including whatever channeling occurred, took some time to form. The initial increasing-rate section in some batch-settling curves was attributed to this.

There has, however, been increasing awareness of the effect of floc structure on thickening. As early as Talmage and Fitch (1955) it was noted,

"The settling velocity of a floc may be presumed to be a function of the structure of the floc, as well as of the solids concentration."

Data were given purporting to show the floc structure varied with initial concentration. In retrospect it seems possible this conclusion was based on an invalid compression model. Michaels and Bolger (1962) show that zone-settling rates were higher after intense prior agitation (Waring blender) than after gentle agitation (upending column). And the effect was reversible! Michaels and Bolger were able to obtain consistent and duplicable results when the same level of prior agitation was maintained, over a limited number of tests in the zone-settling regime. But in recent research by the author, on a different test material and in concentration ranges showing compressive effects, results were far from consistent.

Theories of floc structure, and mechanisms of channeling, will not be further discussed here. It is only noted that the mutability of floc structure can effect and greatly complicate the constitutive relations for compressibility σ and resistivity, k.

DESIGN IMPLICATIONS

The procedures used for sedimentation design may be classified and outlined as follows:

1. Clarification design
   - a. Long tube
   - b. Second order retention tests
2. Zone settling unit area
   - c. Coe and Clevenger tests
   - d. Kynch interpretation of single batch test
   - e. Talmage and Fitch
   - f. Oltmann construction
3. Pinch point in compression
   - g. Coe and Clevenger tests in deep columns
   - h. Kynch interpretation of batch test in deep column
4. Compaction
   - i. Coe and Clevenger compression test
   - j. Michaels and Bolger model
   - k. Kos approximate model

Procedures under headings 1 and 2 are well known, and are fully described in text or reference books such as the cited one edited by Purchas. They will not be further described here. The same is true for the classical Coe and Clevenger compression test (i).

Of these, only the methods for clarification seem uniformly reliable for commercial design. Those for thickening are not. And the reason is that the models on which they are based are not uniformly valid. The remainder of the methods are based on what are hoped to be more valid models, but none are experimentally well-confirmed. We simply do not know, at this time, how widely valid and reliable they will be.

Thickeners have been designed by the classical methods of Coe and Clevenger for well over half a century. Mostly they didn't turn out too badly, but even Coe and Clevenger specifically denied that their method for compression [based on the premiss or model that in compression, $C = C(t)$] was uniformly valid. Their method for determining unit area (item c above) was long considered gospel. But particularly after development of Kynch theory and the Talmage and Fitch design procedure, it became evident that their zone-settling model also was occasionally invalid. Kynch procedures, which were supposedly based on the same model, sometimes predicted unit areas from twice to several times as large as those predicted by Coe and Clevenger tests. And papers started appearing in the literature comparing predictions with actual thickener operation. In some instances, the Coe and Clevenger method underestimated required unit area by as much as fifty percent. The Talmage and Fitch method, on the other hand, seemed consistently to overestimate it. Sometimes the Coe and Clevenger prediction was the closer, sometimes the Talmage and Fitch one. Neither method seemed consistently the more correct.

Two explanations have been suggested for the difference. First, in the Coe and Clevenger method, settling rate at any concentration is determined in a test having that concentration initially. In the Talmage and Fitch method, the higher concentrations develop in a test originally at a lower concentration. Perhaps the floc structure is different in the two cases. Second, it may be that what was conceived to be zone settling was actually in compression, in which case the Kynch assumptions are not met. In that case, there is no apparent theoretical reason why the two types of batch tests would have to give the same results, or that either should predict accurately what would be required in continuous thickening.

The pinch point methods listed under heading 3 above should permit the critical zone to be in the compression regime. They purport to give a lower bound for required unit area, but would not be valid where extra area is needed to satisfy the demands of compaction. They would predict working unit area only if underflow concentration is not too high. The first of them, (item g) is described by Fitch (1966) but is without experimental validation. The second (item h) is given by Wilhelm and Naide (1979) and was validated against continuous operation for at least some cases. Note, however that these methods predict nothing about underflow concentration, or of how deep the compaction zone should be.

Compaction procedures j and k have been suggested (Fitch, 1966 and 1975). How good these prove to be will depend upon the validity of the models from which they were derived. But again, these methods thus far lack experimental validation.

What is badly needed at this time is more validation tests. Predictions of the various methods should be compared to steady-state thickening results, preferably in commercial sized units. Until this is done, there will continue to be much art in the design of thickeners, together with a certain amount of pure faith.

NOTATION

a = acceleration of system
a' = constant in Adorjan equation
A = settling area
b = constant in Adorjan equation
$\hat{c}$ = volume fraction solids, $- C/\rho_s$
C = mass solids concentration
C_a = concentration at point a
C_c = concentration at critical zone
C_0 = initial concentration in batch test
C_u = concentration of thickener underflow
$(dC/dx)^*$ = concentration gradient in batch test after subsidence is complete
D = dilution, wt liquid/wt solids
D_u = dilution in thickener underflow
e = vol liq/vol solids, $= (1 - \hat{c})/\hat{c}$
g = acceleration of gravity
G = mass flux of solids
G_θ = mass flux of solids through steady state thickening
H = height of pulp-supernatant interface in batch settling test
H_i = value of H at Kynch tangent intercept
H_D = characteristic height parameter in Dick equation
H_0 = initial height in batch settling test
j = volume of floccules/volume of solids in floccule
k = resistivity coefficient
K = flocculation rate constant
N = Dixon dimensionless parameter, measuring gravitational force/inertial force
P = fluid pressure
$\overline{P}$ = dynamic fluid pressure, P.-static head
Q_o = overflow rate
Q_u = underflow rate
S = solids flux in absence of solid stress gradient
$\mathbf{S}$ = surface vector
t = time
t_i = value of time at intercept of Kynch tangent
t_u = value of time parameter in Talmage and Fitch construction
u = settling velocity of solids (with respect to pulp)
u_f = velocity of fluid (with respect to pulp)
u_i = initial settling rate in batch test
u_0 = settling rate when $\hat{c} \to 0$
u_* = initial settling rate in batch column of infinite height $= u$ in absence of solids stress gradient
v = velocity of system or pulp as a whole
v_o = overflow rate
Y = Michaels and Bolger yield height parameter
z = depth of impression zone

Greek Letters

α = Kynch propagation velocity of discontinuity
β = Kynch propagation velocity for zone of constant concentration
ψ = solids stress per unit area, solids pressure
ρ_f = density of liquid phase
ρ_s = density of solids phase
σ = compressive yield value of solids structure in compressing pulp
τ = shear yield value of solids structure
ω = volume of solids per unit area above point (Shirato coordinate)

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